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ELASTIC WAVE PROPAGATION IN FILAMENTARY COMPOSITE MATERIALS†

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Abstract—An approximate first order theory for elastic wave propagation in unidirectional, filamentary composite materials is developed. Included are stress equations of motion, boundary conditions and constitutive relations. For waves propagating parallel to the fiber orientation in an extended medium, the motion separates into three distinct types: longitudinal, flexural and torsional. All motions are dispersive and sensitive to changes in relative material stiffnesses and geometry. For propagation perpendicular to the fiber orientation, the motion is dispersive and the frequency spectra show stopping bands typical of periodic media.

1. INTRODUCTION

IN ADDITION to their high strength over extended temperature ranges modern engineering composites possess properties which are potentially important as pulse-attenuation mechanisms. Among these, geometric dispersion, resulting from the interaction of the stress wave with the constituents, can make a significant contribution to the total dispersive nature of the material. The so-called "effective modulus" theories, which adequately describe the static behavior of composites, have been shown to be inadequate for describing the dispersive character of laminated composites [1]. As a result, much recent effort has been directed towards the development of improved theories to describe the dynamic response of these materials.

One of the most active of these programs has resulted in the "effective stiffness theory" [1–9], which has been used extensively for analyzing the gross response of periodically laminated composites. The success of these studies suggests that the same basic approach can be used to describe the dynamic response of unidirectional filamentary composites as well. These materials, consisting of long stiff fibers deliberately oriented in a single direction and embedded in a softer matrix, serve as the basic building blocks of many laminated composites.

In an early paper [3], Achenbach and Herrmann used the effective stiffness theory approach to describe an extended filamentary composite. They found that for waves

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propagating parallel to the fiber orientation, the motion separated into three distinct types: longitudinal, flexural and torsional. Of these, only the flexural motion was found to be dispersive. It was noted that with inclusion of thickness stretch motion, the theory could account for dispersion of longitudinal waves as well. For waves propagating perpendicular to the fiber orientation, all motions were found to be nondispersive. No subsequent improvements or refinements of the model have appeared in the literature.

In a recent report [10], a different model, using the techniques of the effective stiffness theory, was proposed for the filamentary composite. For this theory, the longitudinal motion was shown to be dispersive. However, no numerical results were reported pending experimental determination of certain constants introduced in the development.

In the present paper, a more complex model is employed to describe the unidirectional filamentary composite. The theoretical development closely parallels that of the effective stiffness theory for laminates [6–7] and is properly characterized as an extension of that theory to the more involved geometry of the filamentary composite. The physical model consists of parallel fibers of rectangular cross section embedded in a matrix. It is recognized that few actual composites have rectangular fibers. However, it has been demonstrated that dispersion curves for bars of circular and square cross sections are nearly the same if their cross sectional areas are approximately the same [11]. A similar relationship between dispersion results for composites with square and circular fibers is anticipated. The energy method adopted permits formulation of both equations of motion and boundary conditions. For waves propagating parallel to the fibers in an extended composite material, the theory predicts dispersion of longitudinal, flexural and torsional waves. In addition, for waves propagating perpendicular to the fibers, the motion is also found to be dispersive.

2. THEORY

A cross sectional view of a solid body is depicted in Fig. 1. The body is divided into a large number of bars of rectangular cross section (heavy lines) which extend completely through the body in the direction normal to the page. Each such bar is called a basic cell of

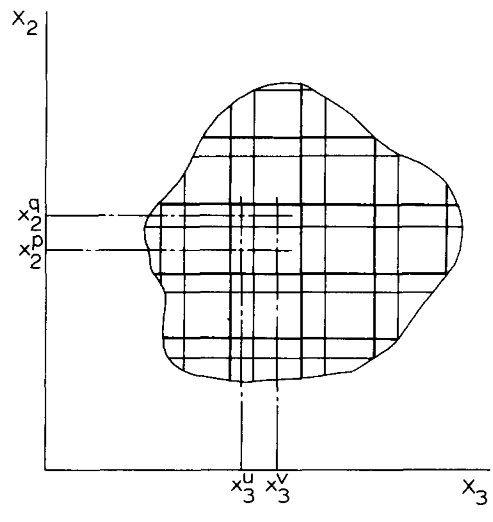

FIG. 1. Cross section of a solid body showing division into basic cells and elements.

the composite. Each basic cell is, in turn, subdivided into four bars of rectangular cross section (light lines). Each subdivision is called an element of the basic cell. Each element is restricted to be composed of a single, homogeneous, isotropic, linearly elastic material so that a basic cell contains at most four different materials. The pattern of any one basic cell is repeated for every other basic cell of the body so that corresponding elements in all cells are necessarily composed of the same material. Figure 2 represents one possible composite which may be so constructed—a filamentary composite.

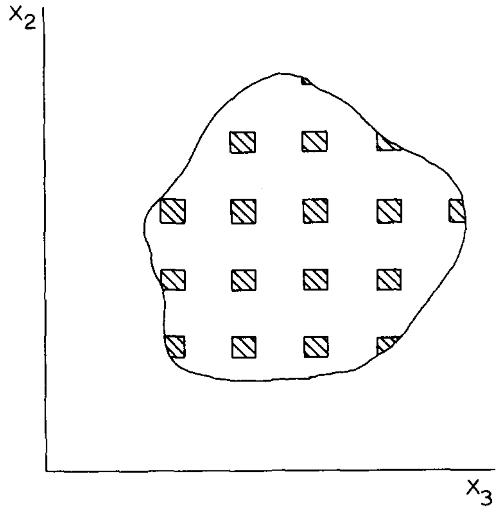

FIG. 2. A filamentary composite.

A reference macrocoordinate system x_1, x_2, x_3 is established and used to locate points on the centerline of each element. Thus, (x_2^q, x_3^u) locates the centerline of the element in the qth row and uth column of the assemblage of elements (referred to as the quth element).

An expanded view of the basic cell located by the macrocoordinates x_2^p, x_2^q, x_3^u and x_3^v is given in Fig. 3. The dimensions of the cell are defined by H_1, H_2, h_1 and h_2. Local coordinates $[\bar{x}_2^r, \bar{x}_3^s(r = p; q, s = u, v)]$ are established on the centerline of each element. The displacement of a point within an element is then a function of the macrocoordinates (locating a point on the centerline), the local coordinates (locating the point relative to the centerline), and the time parameter t.

To arrive at a first order approximate theory, the displacement field within each element is expanded in a Taylor's series about the centerline. The series is then truncated and the first three terms of that series are retained, hence

$$u_j^{rs} = (u_j^0)^{rs} + \bar{x}_2^r \psi_{2j}^{rs} + \bar{x}_3^s \psi_{3j}^{rs} (j = 1, 2, 3; rs = pu, pv, qu, qv) \tag{1}$$

where $(u_j^0)^{rs}$ and ψ_{ij}^{rs} are functions of the macrocoordinates only. Conceptually, this approximation improves as the size of the element becomes small compared to the length characterizing the deformation.

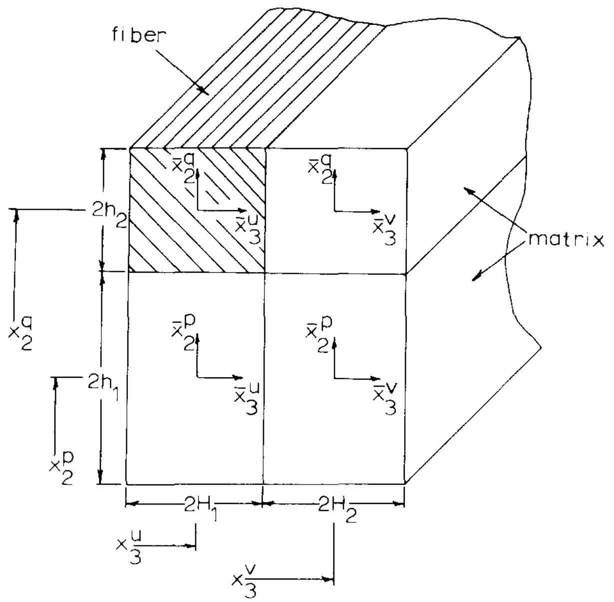

FIG. 3. A portion of a basic cell of a two material composite.

As in the laminate theory [6], dynamic interaction between elements is ensured by matching displacements at the interfaces between elements. For example, at the midpoint of the interface between the puth and quth elements

$$u_j^{pu}\Big|_{\substack{\bar{x}_2^p = h_1 \\ \bar{x}_3^u = 0}} = u_j^{qu}\Big|_{\substack{\bar{x}_2^q = -h_2 \\ \bar{x}_3^u = 0}} . \tag{2}$$

Substituting (1) into (2) results in

$$\frac{(u_j^0)^{qu}(x_1, x_2^p + (h_1+h_2), x_3^u, t) - (u_j^0)^{pu}(x_1, x_2^p, x_3^u, t)}{(h_1+h_2)}$$
$$= \frac{h_1}{h_1+h_2}\psi_{2j}^{pu} + \frac{h_2}{h_1+h_2}\psi_{2j}^{qu} + \frac{\bar{x}_3^u}{h_1+h_2}(\psi_{3j}^{pu} - \psi_{3j}^{qu}). \tag{3}$$

Since a slowly varying displacement field has been assumed, the centerline displacements of all four elements are approximately the same and may be described by a single function u_j^0.

$$(u_j^0)^{pu} \approx (u_j^0)^{pv} \approx (u_j^0)^{qu} \approx (u_j^0)^{qv} \equiv u_j^0. \tag{4}$$

In addition it may be assumed that

$$\begin{aligned} x_2^p &\approx x_2^q \equiv x_2 \\ x_3^u &\approx x_3^v \equiv x_3 \end{aligned} \tag{5}$$

since differences in the values of the functions ψ_{ij}^{rs} and u_j^0 between x_2^p and x_2^q and between x_3^u and x_3^v may be neglected. The displacement functions u_j^0 and ψ_{ij}^{rs} may then be regarded as continuous functions of macrocoordinates, describing the deformation in corresponding elements of every basic cell throughout the body.

With these assumptions, (3) may be replaced by the approximate condition

$$\frac{\partial u_j^0}{\partial x_2} = \eta_1 \psi_{2j}^{pu} + \eta_2 \psi_{2j}^{qu} \qquad (j = 1, 2, 3) \tag{6a}$$

where $\eta_1 = h_1/(h_1+h_2)$ and $\eta_2 = 1-\eta_1$. Matching displacements at the midpoints of the other interfaces results in 9 additional conditions.

$$\frac{\partial u_j^0}{\partial x_2} = \eta_1 \psi_{2j}^{pv} + \eta_2 \psi_{2j}^{qv} \tag{6b}$$

$$\frac{\partial u_j^0}{\partial x_3} = \xi_1 \psi_{3j}^{qu} + \xi_2 \psi_{3j}^{qv} \tag{6c}$$

$$\frac{\partial u_j^0}{\partial x_3} = \xi_1 \psi_{3j}^{pu} + \xi_2 \psi_{3j}^{pv} \tag{6d}$$

where $\xi_1 = H_1/(H_1+H_2)$ and $\xi_2 = 1-\xi_1$.

Other interface relationships are obtained by matching displacements at the common juncture of the four elements. For example, setting

$$u_j^{qu}\Big|_{\substack{\bar{x}_2^q = -h_2 \\ \bar{x}_3^u = H_1}} = u_j^{pu}\Big|_{\substack{\bar{x}_2^p = h_1 \\ \bar{x}_3^u = H_1}} \tag{7}$$

results in

$$\frac{\partial u_j^0}{\partial x_2} = \eta_1 \psi_{2j}^{pu} + \eta_2 \psi_{2j}^{qu} + H_1(\psi_{3j}^{pu} - \psi_{3j}^{qu}). \tag{8}$$

Comparing (8) with (6a) leads directly to the relationship

$$\psi_{3j}^{pu} = \psi_{3j}^{qu} \tag{9a}$$

Similarly, equating u_j^{qv} and u_j^{pv} at the point of common juncture leads to

$$\psi_{3j}^{qv} = \psi_{3j}^{pv}. \tag{9b}$$

In light of (9a) and (9b) it is evident that (6c) and (6d) are equivalent.

The complete set of interface conditions is found through a similar process. The result is a set of 18 independent conditions relating the displacement functions.

$$\begin{aligned}
u_{j,2}^0 &= \eta_1 \psi_{2j}^{pu} + \eta_2 \psi_{2j}^{qv} \\
u_{j,3}^0 &= \xi_1 \psi_{3j}^{pu} + \xi_2 \psi_{3j}^{qv} \\
\psi_{3j}^{qu} &= \psi_{3j}^{pu} \\
\psi_{3j}^{pv} &= \psi_{3j}^{qv} \\
\psi_{2j}^{qu} &= \psi_{2j}^{qv} \\
\psi_{2j}^{pv} &= \psi_{2j}^{pu}
\end{aligned} \tag{10}$$

where $u_{j,2}^0 \equiv \partial u_j^0/\partial x_2$, $\xi_1 = H_1/(H_1+H_2)$ and $\xi_2 = 1-\xi_1$. Since a linear displacement distribution was assumed, matching displacements at two points of every interface ensures that displacements match everywhere along each interface. Treating u_j^0 and ψ_{ij}^{rs} as continuous functions of the macrocoordinates ensures continuity of displacements within every basic cell of the body. Finally, if differences in the value of ψ_{ij}^{rs} for adjacent cells are ignored, the interface conditions (10) ensure matching of displacements between adjacent basic cells throughout the body. With (10), the number of independent unknown functions may be reduced from 27 to 9. For convenience u_j^0, ψ_{2j}^{pu} and ψ_{3j}^{pu} were selected as the independent set.

As in the laminate theory [6], Hamilton's principle will be employed to derive equations of motion and boundary conditions. Mathematically, this is expressed by

$$\delta \int_{t_0}^{t_1} (T-W)\,\mathrm{d}t + \int_{t_0}^{t_1} \delta W_e\,\mathrm{d}t = 0 \tag{11}$$

where T is the total kinetic energy, W is the total strain energy and W_e is the work of the external forces. δ indicates the first variation of the quantities indicated where variations are properly taken on the independent displacement functions u_j^0 and ψ_{ij}^{pu}.

Since each element of the basic cell obeys the laws of classical linear elasticity, the strain energy per unit volume of the element is given by

$$\begin{aligned}\overline{W}^{rs} &= \tfrac{1}{2}(\lambda_{rs}+2\mu_{rs})(e_{11}^{rs^2}+e_{22}^{rs^2}+e_{33}^{rs^2}) \\ &\quad + \lambda_{rs}(e_{11}^{rs}e_{22}^{rs}+e_{11}^{rs}e_{33}^{rs}+e_{22}^{rs}e_{33}^{rs}) + 2\mu_{rs}(e_{12}^{rs^2}+e_{13}^{rs^2}+e_{23}^{rs^2})\end{aligned} \tag{12}$$

where λ_{rs} and μ_{rs} are the Lamé constants for the material of the rs element and e_{ij}^{rs} are the strain components for that element. The strain components are given by

$$e_{ij}^{rs} = \tfrac{1}{2}(u_{j,i}^{rs}+u_{i,j}^{rs}) \tag{13}$$

where u_j^{rs} is given by (1). In (13), partial differentiation is properly done with respect to the local coordinates: i.e. $[\partial(\ )/\partial x_2]$ means $[\partial(\ )/\partial \bar{x}_2^r]$ and $[\partial(\ )/\partial x_3]$ means $[\partial(\ )/\partial \bar{x}_3^s]$ [6].

The total strain energy of the rs element is obtained by integrating (12) over the volume of the element

$$\begin{aligned} W_T^{rs} &= \int_{x_1}\int_{-h}^{h}\int_{-H}^{H} \overline{W}^{rs}\,\mathrm{d}\bar{x}_3^s\,\mathrm{d}\bar{x}_2^r\,\mathrm{d}x_1 \\ &= \int_{x_1} W_c^{rs}\,\mathrm{d}x_1 \qquad (rs = pu, pv, qu, qv).\end{aligned} \tag{14}$$

The total strain energy of the basic cell is obtained by adding the strain energies of the elements

$$W_T^{\text{cell}} = \int_{x_1} (W_c^{pu}+W_c^{pv}+W_c^{qu}+W_c^{qv})\,\mathrm{d}x_1 . \tag{15}$$

Finally, the strain energy/unit cross sectional area is obtained by dividing (15) by the cross sectional area of the cell.

$$W_c = \int_{x_1}\left[\frac{W_c^{pu}+W_c^{pv}+W_c^{qu}+W_c^{qv}}{4(H_1+H_2)(h_1+h_2)}\right]\mathrm{d}x_1 . \tag{16}$$

The integrand of (16) then represents the strain energy/unit volume of the basic cell and hence of the composite body.

Following Achenbach *et al.* [7] and Mindlin [12], strains and strain gradients are defined as

(a) Gross strain:

$$e_{ij}^0 \equiv \tfrac{1}{2}(u_{j,i}^0 + u_{i,j}^0) \qquad (i, j = 1, 2, 3).$$

(b) Relative deformation:

$$\gamma_{ij} \equiv u_{j,i}^0 - \psi_{ij} \qquad (i = 2, 3;\ j = 1, 2, 3). \tag{17}$$

(c) Gradient of relative deformation:

$$\theta_{kij} = \gamma_{ij,k} \qquad (k = 1, i = 2, 3;\ j = 1, 2, 3).$$

(d) Local deformation gradient:

$$\kappa_{kij} = \psi_{ij,k} \qquad (k = 1, i = 2, 3;\ j = 1, 2, 3).$$

where the superscript pu has been dropped from the ψ_{ij} terms for notational simplicity. The computations indicated in (12)–(14) and (16) were carried out to calculate the strain energy/unit volume of the cell, $\overline{W}$. Using the interface conditions (10) to express $\overline{W}$ in terms of the 9 independent displacement functions and subsequently the definitions of (17), the strain energy per unit volume of the composite may be written as a quadratic form

$$\begin{aligned}\overline{W} = {} & A_{ijkl}e_{ij}^0 e_{kl}^0 + B_{ijkl}e_{ij}^0\gamma_{kl} + C_{ijkl}\gamma_{ij}\gamma_{kl} \\ & + D_{ijklmn}\kappa_{ijk}\kappa_{lmn} + E_{ijklmn}\kappa_{ijk}\theta_{lmn} + F_{ijklmn}\theta_{ijk}\theta_{lmn}\end{aligned} \tag{18}$$

where the non-zero constants are given as

$$A_{1111} = A_{2222} = A_{3333} = \tfrac{1}{2}[G_{qu}\eta_2\xi_1 + G_{qv}\eta_2\xi_2 + G_{pu}\eta_1\xi_1 + G_{pv}\eta_1\xi_2]$$

$$\begin{aligned}A_{1122} &= A_{2211} = A_{1133} = A_{3311} \\ &= A_{2233} = A_{3322} = \tfrac{1}{2}[\lambda_{qu}\eta_2\xi_1 + \lambda_{qv}\eta_2\xi_2 + \lambda_{pu}\eta_1\xi_1 + \lambda_{pv}\eta_1\xi_2]\end{aligned}$$

$$\begin{aligned}A_{1212} &= A_{2112} = A_{1221} = A_{2121} = A_{1313} \\ &= A_{1331} = A_{3113} = A_{3131} = A_{2323} \\ &= A_{3223} = A_{2332} = A_{3232} = \tfrac{1}{2}[\mu_{qu}\eta_2\xi_1 + \mu_{qv}\eta_2\xi_2 + \mu_{pu}\eta_1\xi_1 + \mu_{pv}\eta_1\xi_2]\end{aligned}$$

$$B_{2222} = G_{qu}\eta_1\xi_1 + G_{qv}\eta_1\xi_2 - G_{pu}\eta_1\xi_1 - G_{pv}\eta_1\xi_2$$

$$B_{3333} = -G_{qu}\eta_2\xi_1 + G_{qv}\eta_2\xi_1 - G_{pu}\eta_1\xi_1 + G_{pv}\eta_1\xi_1$$

$$B_{1122} = B_{3322} = \lambda_{qu}\eta_1\xi_1 + \lambda_{qv}\eta_1\xi_2 - \lambda_{pu}\eta_1\xi_1 - \lambda_{pv}\eta_1\xi_2$$

$$B_{1133} = B_{2233} = -\lambda_{qu}\eta_2\xi_1 + \lambda_{qv}\eta_2\xi_1 - \lambda_{pu}\eta_1\xi_1 + \lambda_{pv}\eta_1\xi_1$$

$$B_{1221} = B_{2121} = B_{2323} = B_{3223} = \mu_{qu}\eta_1\xi_1 + \mu_{qv}\eta_1\xi_2 - \mu_{pu}\eta_1\xi_1 - \mu_{pv}\eta_1\xi_2$$

$$B_{1331} = B_{3131} = B_{2332} = B_{3232} = -\mu_{qu}\eta_2\xi_1 + \mu_{qv}\eta_2\xi_1 - \mu_{pu}\eta_1\xi_1 + \mu_{pv}\eta_1\xi_1$$

$$C_{2222} = \tfrac{1}{2}[G_{qu}\xi_1 + G_{qv}\xi_2)\eta_1^2/\eta_2 + (G_{pu}\xi_1 + G_{pv}\xi_2)\eta_1]$$
$$C_{3333} = \tfrac{1}{2}[(G_{qu}\eta_2 + G_{pu}\eta_1)\xi_1 + (G_{qv}\eta_2 + G_{pv}\eta_1)\xi_1^2/\xi_2]$$
$$C_{3232} = C_{3131} = \tfrac{1}{2}[(\mu_{qu}\eta_2 + \mu_{pu}\eta_1)\xi_1 + (\mu_{qv}\eta_2 + \mu_{pv}\eta_1)\xi_1^2/\xi_2]$$
$$C_{2323} = C_{2121} = \tfrac{1}{2}[(\mu_{qu}\xi_1 + \mu_{qv}\xi_2)\eta_1^2/\eta_2 + (\mu_{pu}\xi_1 + \mu_{pv}\xi_2)\eta_1]$$
$$C_{2332} = C_{3223} = \tfrac{1}{2}[-\mu_{qu} + \mu_{qv} + \mu_{pu} - \mu_{pv}]\eta_1\xi_1$$
$$C_{2233} = C_{3322} = \tfrac{1}{2}[-\lambda_{qu} + \lambda_{qv} + \lambda_{pu} - \lambda_{pv}]\eta_1\xi_1$$

$$D_{121121} = \tfrac{1}{6}[(G_{qu}\xi_1 + G_{qv}\xi_2)\eta_2 h_2^2 + (G_{pu}\xi_1 + G_{pv}\xi_2)\eta_1 h_1^2]$$
$$D_{131131} = \tfrac{1}{6}[(G_{qu}\eta_2 + G_{pu}\eta_1)\xi_1 H_1^2 + (G_{qv}\eta_2 + G_{pv}\eta_1)\xi_2 H_2^2]$$
$$D_{122122} = D_{123123} = \tfrac{1}{6}[(\mu_{qu}\xi_1 + \mu_{qv}\xi_2)\eta_2 h_2^2 + (\mu_{pu}\xi_1 + \mu_{pv}\xi_2)\eta_1 h_1^2]$$
$$D_{132132} = D_{133133} = \tfrac{1}{6}[(\mu_{qu}\eta_2 + \mu_{pu}\eta_1)\xi_1 H_1^2 + (\mu_{qv}\eta_2 + \mu_{pv}\eta_1)\xi_2 H_2^2]$$

$$E_{121121} = \tfrac{1}{3}[G_{qu}\xi_1 + G_{qv}\xi_2]h_2^2$$
$$E_{131131} = \tfrac{1}{3}[G_{qv}\eta_2 + G_{pv}\eta_1]H_2^2$$
$$E_{132132} = E_{133133} = \tfrac{1}{3}[\mu_{qv}\eta_2 + \mu_{pv}\eta_1]H_2^2$$
$$E_{122122} = E_{123123} = \tfrac{1}{3}[\mu_{qu}\xi_1 + \mu_{qv}\xi_2]h_2^2$$

$$F_{121121} = \tfrac{1}{6}[G_{qu}\xi_1 + G_{qv}\xi_2]h_2^2/\eta_2$$
$$F_{131131} = \tfrac{1}{6}[G_{qv}\eta_2 + G_{pv}\eta_1]H_2^2/\xi_2$$
$$F_{122122} = F_{123123} = \tfrac{1}{6}[\mu_{qu}\xi_1 + \mu_{qv}\xi_2]h_2^2/\eta_2$$
$$F_{132132} = F_{133133} = \tfrac{1}{6}[\mu_{qv}\eta_2 + \mu_{pv}\eta_1]H_2^2/\xi_2$$

$$G_{qu} = \lambda_{qu} + 2\mu_{qu}$$
$$G_{qv} = \lambda_{qv} + 2\mu_{qv}$$
$$G_{pu} = \lambda_{pu} + 2\mu_{pu}$$
$$G_{pv} = \lambda_{pv} + 2\mu_{pv}.$$

As in the laminate theory [7], stresses are defined in terms of derivatives of the strain energy density.

(a) Cauchy stress:

$$\tau_{ij} = \tau_{ji} \equiv \frac{\partial \overline{W}}{\partial e_{ij}^0} \qquad (i,j = 1,2,3)$$

(b) Relative stress:

$$\sigma_{ij} = \frac{\partial \overline{W}}{\partial \gamma_{ij}} \qquad (i = 2,3;\, j = 1,2,3)$$

(c) Double stress:

$$\mu_{kij} = \frac{\partial \overline{W}}{\partial \kappa_{kij}}$$

$$\chi_{kij} = \frac{\partial \overline{W}}{\partial \theta_{kij}} \qquad (k = 1;\, i = 2,3;\, j = 1,2,3) \tag{19}$$

with these definitions and the quadratic form of (18), the variation of the total strain energy may be computed. The procedure and results are described in [7].

An expression for the kinetic energy density is obtained in the same manner used to compute the strain energy density

$$\overline{T} = \frac{T_c^{qu} + T_c^{qv} + T_c^{pu} + T_c^{pv}}{4(H_1 + H_2)(h_1 + h_2)} \tag{20}$$

where

$$T_c^{rs} = \int_{-H}^{H} \int_{-h}^{h} \tfrac{1}{2}\rho_{rs} \dot{u}_j^{rs} \dot{u}_j^{rs} \, \mathrm{d}\bar{x}_2^r \, \mathrm{d}\bar{x}_3^s.$$

ρ_{rs} is the density of the material of the rs element and $\dot{u}_j^{rs} = \partial u_j^{rs}/\partial t$. The resulting expression for the kinetic energy of the composite is

$$\overline{T} = M_{ij}\dot{u}_i^0 \dot{u}_j^0 + N_{ijkl}\dot{u}_{i,j}^0 \dot{u}_{k,l}^0 + P_{ijkl}\dot{\psi}_{ij}\dot{\psi}_{kl} + Q_{ijkl}\dot{u}_{i,j}^0 \dot{\psi}_{kl} \tag{21}$$

where the non-zero constants are given by

$$M_{11} = M_{22} = M_{33} = [(\rho_{qu}\xi_1 + \rho_{qv}\xi_2)\eta_2 + (\rho_{pu}\xi_1 + \rho_{pv}\xi_2)\eta_1]/2$$

$$N_{1212} = N_{2222} = N_{3232} = [\rho_{qu}\xi_1 + \rho_{qv}\xi_2]h_2^2/6\eta_2$$

$$N_{1313} = N_{2323} = N_{3333} = [\rho_{pv}\eta_1 + \rho_{qv}\eta_2]H_2^2/6\xi_2$$

$$P_{2121} = P_{2222} = P_{2323} = [(\rho_{qu}\xi_1 + \rho_{qv}\xi_2)\eta_1^2 h_2^2/\eta_2 + (\rho_{pv}\xi_2 + \rho_{pu}\xi_1)\eta_1 h_1^2]/6$$

$$P_{3131} = P_{3232} = P_{3333} = [(\rho_{qu}\eta_2 + \rho_{pu}\eta_1)\xi_1 H_1^2 + (\rho_{pv}\eta_1 + \rho_{qv}\eta_2)\xi_1^2 H_2^2/\xi_2]/6$$

$$Q_{1221} = Q_{2222} = Q_{3223} = -[\rho_{qu}\xi_1 + \rho_{qv}\xi_2]\eta_1 h_2^2/3\eta_2$$

$$Q_{1331} = Q_{2332} = Q_{3333} = -[\rho_{pv}\eta_1 + \rho_{qv}\eta_2]\xi_1 H_2^2/3\xi_2.$$

Again, the variation of the total kinetic energy is obtained as in [7].

Following Mindlin [12], the virtual work of the external forces is postulated as

$$\delta W_e = \int_v (f_j \delta u_j^0 + F_{ij}\delta\psi_{ij})\, \mathrm{d}V + \int_S (t_j \delta u_j^0 + T_{ij}\delta\psi_{ij} + Q_j D\delta u_j^0)\, \mathrm{d}S + \oint_c p_j \delta u_j^0 \, \mathrm{d}s \tag{22}$$

where f_j, F_{ij}, t_j, T_{ij} are components of body forces/unit volume, body double forces/unit volume, surface forces/unit area and surface double forces/unit area, respectively. Q_j and p_j may be interpreted as linear combinations of surface double force components per unit area. D is an operator, $D = n_k[\partial(\ )/\partial x_k]$ and n_k are components of the outward unit normal to the bounding surface.

Applying Hamilton's Principle then leads to the governing equations of motion and boundary conditions for the theory.

In V:

$$\tau_{ij,i}+\sigma_{ij,i}-\chi_{kij,ki}-2M_{jl}\ddot{u}_l^0 + 2N_{jknp}\ddot{u}^0_{n,pk}+Q_{jkpl}\ddot{\psi}_{pl,k}+f_j = 0 \tag{23a}$$

$$\mu_{kij,k}-\chi_{kij,k}+\sigma_{ij}-2P_{npij}\ddot{\psi}_{np}-Q_{lmij}\ddot{u}^0_{l,m}+F_{ij} = 0 \tag{23b}$$

on S:

$$-n_i(\tau_{ij}+\sigma_{ij}-\chi_{kij,k})-(D_rn_r)n_kn_i\chi_{kij}+n_kD_i\chi_{kij}+(D_in_k)\chi_{kij} + t_j-n_k(2N_{jknp}\ddot{u}^0_{n,p}+Q_{jkpl}\ddot{\psi}_{pl}) = 0 \tag{24a}$$

$$-n_k(\mu_{kij}-\chi_{kij})+T_{ij} = 0 \tag{24b}$$

$$-n_in_k\chi_{kij}+Q_j = 0 \tag{24c}$$

on C:

$$-[n_k\varepsilon_{rti}s_rn_t\chi_{kij}]+p_j = 0 \tag{25}$$

where $D_i = (\delta_{ki}-n_in_k)[\partial(\ )/\partial x_k]$ and ε_{rti} is the alternator tensor. In (25) the brackets ([ ]) indicate the difference in the enclosed quantity as the bounding curve C is approached from different sides. For a body with an edge this quantity will, in general, be non-zero. It is evident that although the stresses, tractions, body forces, etc. of the present theory are defined differently, the equations of motion and boundary conditions have the same form as those of the laminate theory [7].

If all four elements are taken to be the same material and the limiting case $H_1 \to H_2 \to h_1 \to h_2 \to 0$ is considered (while $\eta_1, \eta_2, \xi_1, \xi_2$ remain bounded), the equations of motion and boundary conditions return to those of classical linear elasticity. In that limit, the motion of a point (basic cell) of the body is completely described by the gross displacement u_j^0.

If the qth row elements are taken to be one material and the pth row elements a different material, the theory describes a laminated composite. The theory is not the same as that of [7], however, and is, in general, less accurate since displacements were approximated in both the x_2 and x_3 directions in the present paper. Only in the limit as $H_1 \to H_2 \to 0$ where ξ_1 and ξ_2 remain bounded, with

$$\psi_{ij}^{qu} = \psi_{ij}^{qv} = \psi_{ij}^{m}$$

$$\psi_{ij}^{pu} = \psi_{ij}^{pv} = \psi_{ij}^{f}$$

and for the case of plane strain in the x_3 direction ($u_3^0 = \psi_{23} = \psi_{33} = 0$ and for no dependence on the x_3 coordinate) are the theories the same.

In [9], an effective stiffness theory for layered composite cylinders was formulated in terms of averaged stresses and moments of stresses. That study suggests a method for interpreting the stresses and double stresses of the present theory. The elastic strains for each element may be calculated from (13). The stresses in the rs element may then be calculated from the generalized Hooke's law

$$\sigma_{ij}^{rs} = \lambda_{rs} e_{kk}^{rs} \delta_{ij} + 2\mu_{rs} e_{ij}^{rs} \tag{26}$$

where λ_{rs}, μ_{rs} are the Lamé constants, δ_{ij} is the Kronecker delta and σ_{ij}^{rs} are the stress components for the rs element. Average stresses and moments of stresses may then be defined.

$$\bar{\sigma}_{ij}^{rs} = \frac{\int_{-H}^{H}\int_{-h}^{h} \sigma_{ij}^{rs}\, d\bar{x}_2^r\, d\bar{x}_3^s}{4(H_1+H_2)(h_1+h_2)} \tag{27}$$

$$\overline{m}_{2ij}^{rs} = \frac{\int_{-H}^{H}\int_{-h}^{h} \sigma_{ij}^{rs}\bar{x}_2^r\, d\bar{x}_2^r\, d\bar{x}_3^s}{4(H_1+H_2)(h_1+h_2)} \tag{28}$$

$$\overline{m}_{3ij}^{rs} = \frac{\int_{-H}^{H}\int_{-h}^{h} \sigma_{ij}^{rs}\bar{x}_3^s\, d\bar{x}_2^r\, d\bar{x}_3^s}{4(H_1+H_2)(h_1+h_2)}. \tag{29}$$

Using the constitutive relations (19), it follows that

$$\begin{aligned}
\tau_{ij} &= \tau_{ji} = \bar{\sigma}_{ij}^{qu} + \bar{\sigma}_{ij}^{qv} + \bar{\sigma}_{ij}^{pu} + \bar{\sigma}_{ij}^{pv} \\
\sigma_{2j} &= \frac{\eta_1}{\eta_2}(\bar{\sigma}_{2j}^{qu} + \bar{\sigma}_{2j}^{qv}) - (\bar{\sigma}_{2j}^{pu} + \bar{\sigma}_{2j}^{pv}) \\
\sigma_{3j} &= \frac{\xi_1}{\xi_2}(\bar{\sigma}_{3j}^{pv} + \bar{\sigma}_{3j}^{qv}) - (\bar{\sigma}_{3j}^{pu} + \bar{\sigma}_{3j}^{qu}) \\
\mu_{kij} &= \overline{m}_{ikj}^{pu} + \overline{m}_{ikj}^{pv} + \overline{m}_{ikj}^{qu} + \overline{m}_{ikj}^{qv} \\
\chi_{k2j} &= \frac{1}{\eta_2}(\overline{m}_{2kj}^{qu} + \overline{m}_{2kj}^{qv}) \\
\chi_{k3j} &= \frac{1}{\xi_2}(\overline{m}_{3kj}^{qv} + \overline{m}_{3kj}^{pv}).
\end{aligned} \tag{30}$$

The interpretation of the indices of the double stresses as given by Mindlin [12] and Achenbach *et al.* [7], is thus verified.

These relationships also make possible the interpretation of boundary conditions as averaged forms of Cauchy's law

$$t_j^* = \sigma_{ij}^* n_i \tag{31}$$

where t_j^*, σ_{ij}^* are classical traction and stress components, respectively, and n_i are components of the outward unit normal to the bounding surface. For example, for a plane perperpendicular to the x_1 axis [$\mathbf{n} = (1, 0, 0)$], conditions (24a) and (24b) are non-trivial. From (24b)

$$\begin{aligned}
T_{2j} &= n_1(\mu_{12j} - \chi_{12j}) \\
&= \frac{n_1}{4(H_1+H_2)(h_1+h_2)}\left[\iint_{pu} (\sigma_{1j}^{*pu}\bar{x}_2^p) + \iint_{pv} (\sigma_{1j}^{*pv}\bar{x}_2^p) \right. \\
&\quad \left. -\frac{\eta_1}{\eta_2}\left(\iint_{qu} (\sigma_{1j}^{*qu}\bar{x}_2^q) + \iint_{qv} (\sigma_{1j}^{*qv}\bar{x}_2^q)\right)\right]
\end{aligned} \tag{32}$$

where $\iint_{rs}(\ ) = \int_{-H}^{H}\int_{-h}^{h}(\ )\,d\bar{x}_2^r\,d\bar{x}_3^s$ and (30) has been employed. Comparing (31) and (32) suggests that

$$T_{2j} = (m_{2j}^{pu}+m_{2j}^{pv})-\frac{\eta_1}{\eta_2}(m_{2j}^{qu}+m_{2j}^{qv}) \tag{33}$$

where

$$m_{2j}^{rs} = \frac{\int_{-H}^{H}\int_{-h}^{h} t_j^{*rs}\bar{x}_2^r\,d\bar{x}_2^r\,d\bar{x}_3^s}{4(H_1+H_2)(h_1+h_2)}.$$

These relationships are the same as would be obtained using the smoothing relationship established in [7]. Similar interpretations may be made for the other boundary conditions.

3. WAVE PROPAGATION PARALLEL TO FIBERS

An extended homogeneous body is non-dispersive. An extended composite body, on the other hand, is inherently dispersive because of its internal structure. To investigate this dispersion, we assume travelling waves

$$\left.\begin{aligned} u_m^0 &= C_m \exp\left[\frac{j}{H}(Kx_1-\Omega V_s t)\right] \\ \psi_{lm} &= j\frac{D_{lm}}{H}\exp\left[\frac{j}{H}(Kx_1-\Omega V_s t)\right]\end{aligned}\right\} \quad (l = 2,3;\ m = 1,2,3) \tag{34}$$

where K is the dimensionless wave number, Ω the dimensionless frequency, V_s the shear velocity for the puth element ($V_s^2 = \mu_{pu}/\rho_{pu}$), $j = \sqrt{-1}$ and H is a length parameter.

$$\left.\begin{aligned} H_i &= Hr_i \\ h_i &= Hr_{i+2}\end{aligned}\right\} \quad (i = 1,2) \tag{35}$$

and r_1, r_2, r_3 and r_4 are dimensionless ratios. Substituting (34) into (23) results in four sets of coupled algebraic equations in the amplitudes C_m and D_{lm}. The necessary and sufficient condition for a nontrivial solution of each set of equations is that the determinant of coefficients vanish identically.

The first set of equations describes longitudinal modes of motion consisting of a uniform translation of an element cross section in the x_1 direction, coupled with lateral expansions (contractions) of the section. The frequency determinant is given by

$$\begin{vmatrix} 2(B_1\Omega^2-A_1K^2) & -A_{14}K & -A_{15}K \\ A_{14}K & 2(A_{31}K^2+A_9-B_4\Omega^2) & A_{13} \\ A_{15}K & A_{13} & 2(A_{11}+A_{33}K^2-B_5\Omega^2)\end{vmatrix} = 0 \tag{36}$$

where

$$\begin{aligned} A_1\mu_{pu} &= A_{1111} & A_{31}\mu_{pu}H^2 &= D_{122122}-E_{122122}+F_{122122} \\ A_9\mu_{pu} &= C_{2222} & A_{33}\mu_{pu}H^2 &= D_{133133}-E_{133133}+F_{133133} \\ A_{11}\mu_{pu} &= C_{3333} & B_1\rho_{pu} &= M_{11} \\ A_{13}\mu_{pu} &= 2C_{2233} & B_4\rho_{pu}H^2 &= P_{2222}\end{aligned}$$

$$A_{14}\mu_{pu} = -B_{1122} \qquad B_5\rho_{pu}H^2 = P_{3333}.$$
$$A_{15}\mu_{pu} = -B_{1133}$$

Numerical results obtained by expanding (36) are presented in Fig. 4.

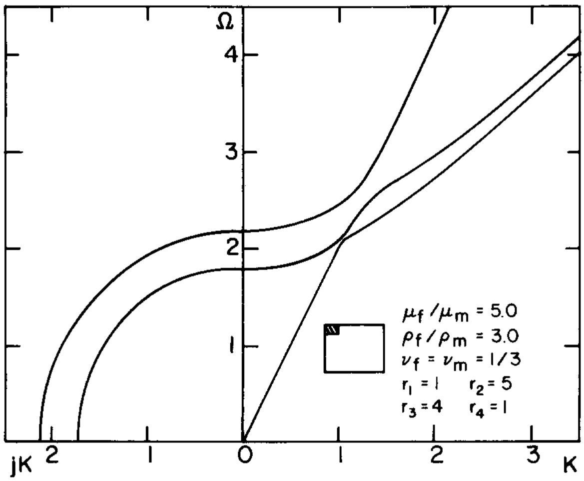

FIG. 4. Frequency spectrum for longitudinal motion.

The second and third sets of equations describe flexural motion consisting of a uniform lateral translation of a section coupled with a rotation about the local axis. For flexure about the $\bar{x}_3$ axis, the frequency equation is

$$\begin{vmatrix} 2(B_1\Omega^2 - A_{24}K^2) & -A_{37}K \\ A_{37}K & 2(A_{30} + A_{10}K^2 - B_4\Omega^2) \end{vmatrix} = 0 \tag{37}$$

where

$$A_{24}\mu_{pu} = A_{1212} \qquad A_{30}\mu_{pu} = C_{2323}$$
$$A_{37}\mu_{pu} = -B_{1221} \qquad A_{10}\mu_{pu}H^2 = D_{121121} - E_{121121} + F_{121121}.$$

Numerical results for flexure are given in Fig. 5.

The fourth set of equations describes torsional motion of an element. The frequency equation is

$$\begin{vmatrix} A_{34} & 2(A_{32} + A_{33}K^2 - B_5\Omega^2) \\ 2(A_{30} + A_{31}K^2 - B_4\Omega^2) & A_{34} \end{vmatrix} = 0 \tag{38}$$

where

$$A_{34}\mu_{pu} = 2C_{2332} \qquad A_{32}\mu_{pu} = C_{3232}.$$

Dispersion results for torsional motion are presented in Fig. 6.

In Figs. 4–6, a cross sectional view of a typical basic cell is depicted in the legend. The quth element was taken as the fiber element, while the remaining three elements were assigned the properties of the matrix. Frequency values corresponding to real values of

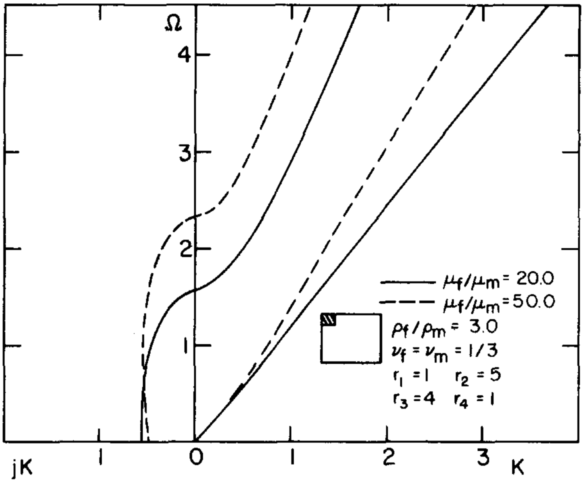

FIG. 5. Frequency spectra for flexural motion.

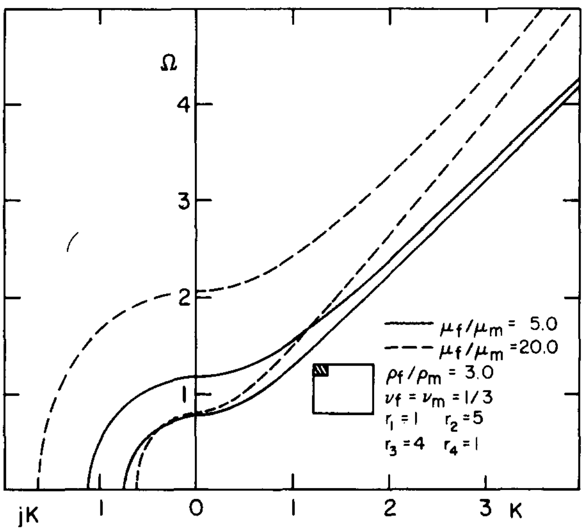

FIG. 6. Frequency spectra for torsional motion.

the wave number (K axis) lie to the right of the origin, while imaginary values of wave number (jK axis) lie to the left. The shear moduli, mass densities, and Poisson's ratios of the fiber and matrix are designated by μ_f, ρ_f, ν_f and μ_m, ρ_m, ν_m, respectively.

Symmetry considerations dictate that the motion in all basic cells of an extended, periodic, filamentary composite is the same. Hence, it may be anticipated that the frequency spectra resulting from the expansion of (36)–(38) should resemble the corresponding spectra for traction free bars, especially when the fiber element is much stiffer than the surrounding matrix.

Typical results for longitudinal motion (Fig. 4), are dispersive and strongly resemble those for rectangular bars [13]. Calculations were carried out for several values of the stiffness ratio (μ_f/μ_m), Poisson's ratio (ν_f) and for several element geometries. Larger stiffness ratios resulted in increased dispersion. Results were also affected by changes in geometry, depending not only on the amount of fiber present, but also on the precise geometric configuration. The dispersion curves showed little sensitivity to changes in Poisson's ratio.

Frequency spectra for flexural motion generally show the same sensitivity to material and geometric parameters described above. Typical results for two values of the stiffness ratio, μ_f/μ_m, are depicted in Fig. 5. The dispersive character of the lower branch increases as the stiffness ratio is increased. In the limiting case ($\mu_f/\mu_m \to \infty$), the slope of the lower branch approaches zero at vanishing wave number and looping between the first and second branches in the imaginary plane occurs. In this limit, the behavior strongly resembles that of bar theories [14].

Results for torsional motion for two values of the stiffness ratio, μ_f/μ_m, are given in Fig. 6. As in the case of flexural motion, the dispersion curves for torsion more closely resemble results for free bars [15] when the ratio of stiffness is increased. The behavior for variations of the other material and geometric parameters is the same as described for longitudinal motion.

4. WAVE PROPAGATION PERPENDICULAR TO FIBERS

The propagation of waves perpendicular to the fiber orientation may be studied by assuming displacements of the form

$$(u_j^0, \psi_{ij}^{rs}) = (A_j, B_{ij}^{rs}) \exp\left[\frac{j}{H}(Kx_3 - \Omega V_s t)\right]. \tag{39}$$

The equations of motion again separate into four sets of coupled equations, three of which are dispersive. The nondispersive motion is that associated with the ψ_{21} displacement function. For that mode, the cross section of each element rotates about an axis parallel to the direction of propagation.

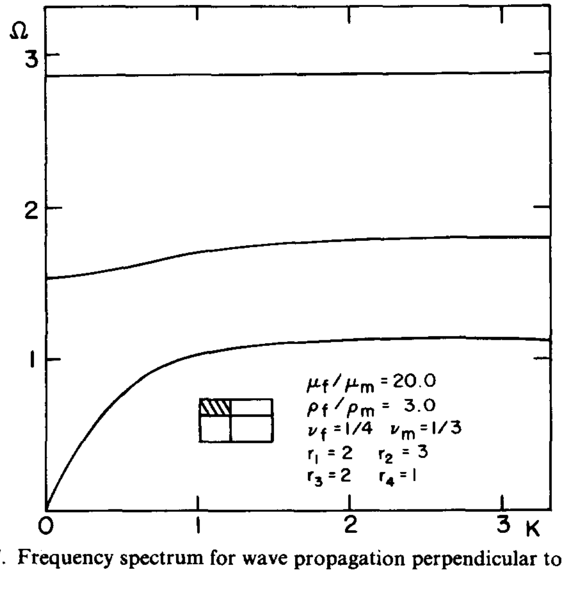

FIG. 7. Frequency spectrum for wave propagation perpendicular to fibers.

Numerical results were obtained for the set of equations which describe a uniform translation of an element section in the x_3 direction coupled with thickness stretch motions. The resulting dispersion relationship is shown in Fig. 7. There are substantial differences between this spectrum and those shown earlier. Stopping bands characteristics of waves in periodic media [16] are evident. In addition, all branches rapidly approach zero group velocity asymptotically. However, since the theory, by its very construction, was designed to describe motion parallel to the fiber, these results should be regarded as valid over a limited range of wave numbers.

5. SUMMARY AND CONCLUSIONS

In the present paper, a first order theory for wave propagation in a unidirectional, filamentary composite was developed. It was demonstrated that the theory predicts dispersion of elastic waves propagating parallel to the fiber orientation and that the frequency spectra for these motions resemble results obtained from bar theories. For the case investigated, wave motion perpendicular to the fiber orientation was also dispersive and exhibited stopping bands typical of periodic media.

As in the laminate theory upon which the present study was based, it is anticipated that the addition of adjustment factors in the strain energy should improve these results. It may also be desirable to include higher order terms in the displacement expansions (1). Results for both the laminate theory and for bar theories indicate that this may be the case. Such judgements await sufficient experimental data since no exact elasticity solutions exist for comparison purposes. The uncorrected theory presented herein should serve as a foundation upon which a more accurate and complete theory may be built.

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Абстракт—Определяется приближенная теория первого порядка для распространения упругой волны в однонаправленных, нитеобразных, составных материалах. Учитываются уравнения движения для напряжений, граничные условия и конститутивные зависимости. Для волн, распространяющихся параллельно к направлению волокна в растягивающей среле, движение оказывается трех разных типов: продольное, изгибное и крутильное. Все движения являются с рассеянием и чувствительные к зменам жесткости и геометрии относительного материала. Для распространения перпендикулярного к направлению волокна, движение есть с рассеянием. Спектры частоты указывают полосы остановки, типичные для периодических сред.